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## Pricing index options by static hedging under finite liquidity

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We develop a model for indifference pricing in derivatives markets where price quotes have bid-ask spreads and finite quantities. The model quantifies the dependence of the prices and hedging portfolios on an investors beliefs, risk preferences and financial position as well as on the price quotes. Computational techniques of convex optimisation allow for fast computation of the hedging portfolios and prices as well as sensitivities with respect to various model parameters. We illustrate the techniques by pricing and hedging of exotic derivatives on S&P index using call and put options, forward contracts and cash as the hedging instruments. The optimized static hedges provide good approximations of the options payouts and the spreads between indifference selling and buying prices are quite narrow as compared with the spread between superhedging and subhedging prices.

*Keywords:* Incomplete markets; indifference pricing; convex optimization.

### 1. Introduction

In incomplete markets, the prices of financial products offered by an agent depend on subjective factors such as views on the future development of the underlying risk factors, risk preferences, the financial position as well as the trading expertise of the agent. An agent's prices also depend on the prices at which the agent can trade other financial products since that affects the costs of (partial) hedging of a product.

The *indifference pricing* principle provides a consistent way to incorporate the

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above factors into a pricing model. A classical reference on indifference pricing of contingent claims under transaction costs is (HN89). In the insurance sector, where market completeness would be quite an unrealistic assumption, indifference pricing seems to have longer history; see e.g. (Büh70). A more recent account with further references can be found in (Car09). The basic theory of indifference pricing was extended to general illiquid market models in (Pen14). The present paper presents a specialized application to the pricing of European style index options in the presence of bid-ask spreads and finite liquidity on the best price quotes.

Indifference pricing builds on an optimal investment model that describes the relevant sector of financial markets as well as the agent's financial position, views and risk preferences. Realistic models are often difficult to solve much like the investment problem they describe. This paper develops a computational framework for indifference pricing of European style options on the S&P500 index. Instead of the usual dynamic trading of the index and a cash-account, we take index options as the hedging instruments. To simplify the modelling and the computations, we consider only buy-and-hold strategies in the options but we take actual market quotes as the trading costs. For the nearest maturities, there are some 200 strikes with fairly liquid quotes. This results in a convex stochastic optimization problem only one risk factor but over 400 decision variables. The model is solved numerically using integration quadratures and an interior point solver for convex optimization. The indifference prices for a given payout are found within seconds so it is easy to study the effect of an agent's views, risk preferences and financial position on the indifference prices.

Much like the Breeden-Litzenberger formula (BL78), indifference pricing provides automatic calibration to quoted option prices. While the Breeden-Litzenberger formula provides only a heuristic approximation in real markets with only a finite number of strikes and finite liquidity, the indifference approach finds the best static hedge given the quotes, the agents views and preferences. Moreover, the indifference approach gives explicit control of the hedging error in incomplete markets. Unlike the Breeden-Litzenberger formula would suggest, we find that in the presence of bid-ask spreads, the optimal hedges are often quite compressed portfolios of options taking positions only in few of the strikes. This is a significant benefit when implementing the hedges in practice. While the Breeden-Litzenberger formula applies only to options whose payouts are differences of convex functions of the underlying, our computational model applies just as well to discontinuous payoffs such as digital options.

## 2. The market

We study exchange traded contingent claims with common maturity  $T$  and payouts that only depend on the value of the S&P500 index at  $T$ . This includes put and call options, forward contracts and cash. In general, lending and borrowing rates for cash are different, so the payout on cash depends nonlinearly on the position

taken. Similarly, the forward rates available in the market depend on whether one takes a long or short position. For the options, on the other hand, the payout per unit held is independent of the position. The payoffs for holding  $x \in \mathbb{R}$  units of an asset are given in Table 1.

Asset	Payoff as a function of the position $x$
Cash	$\min\{e^{r^a T}x, e^{r^b T}x\}$
Forward	$\min\{(X_T - K^a)x, (X_T - K^b)x\}$
Call	$\max\{(X_T - K), 0\}x$
Put	$\max\{(K - X_T), 0\}x$

Table 1. The payoffs as functions of the number of units  $x$  held. Here  $r^a$  and  $r^b$  are the borrowing and lending rates, respectively,  $X_T$  is the value of the underlying at maturity,  $K^a$  and  $K^b$  are forward prices for long and short positions, respectively and  $K$  is the strike price of an option

While the option payoffs are linear in the position, the cost of entering a position depends nonlinearly on the units  $x$ . For a long position  $x > 0$ , one pays the ask-price while for short position, one gets the bid-price. The cost of buying  $x$  units of cash is simply  $x$  while for the forward, the cost is zero.

For each contract, the market quotes come with finite quantities. For the nearest maturity, one can find quotes for some 400 put and call options on S&P500. Table 2 gives an example of quotes available on the 8 April 2016 at 14:55:00 for contracts expiring on 17 June 2016.

Ticker	Type	Bid quantity	Bid price	Ask price	Ask quantity
ESM6 Index	Forward	258	2048.75	2049	377
SPX US 6/17/2016 C2095 Index	Call	623	26.90	28.20	506
SPX US 6/17/2016 P2095 Index	Put	27	72.60	74.70	22

Table 2. Market quotes on 8 April 2016 at 14:55:00 for the forward, a call and a put option maturing 17 June 2016. For the forward, the bid and ask price quotes are the forward prices for entering a short or a long position, respectively. The data was extracted from Bloomberg.

### 3. The portfolio optimisation model

For given initial wealth and quotes on cash, forward and the options, our aim is to find a portfolio with optimal net payoff at maturity. In general, the payoff will depend on the value of the underlying at maturity so the optimality will depend on our risk preferences concerning the uncertain payoffs. The optimality of a portfolio also depends on our financial position which may involve uncertain cash-flows at time  $T$ .

We shall denote our initial wealth by  $w \in \mathbb{R}$  and assume that our financial position obligates us to pay  $c$  units of cash at time  $T$ . The collection of all traded assets (cash, forward, options) is denoted by  $J$ . The cost of buying  $x^j$  units of asset

$j \in J$  will be denoted by

$$S_0^j(x^j) := \begin{cases} s_a^j x^j & \text{if } x^j \geq 0, \\ s_b^j x^j & \text{if } x^j \leq 0, \end{cases} \quad (3.1)$$

where  $s_b^j \leq s_a^j$  are the bid and ask prices of  $j$ . If  $j$  is cash, we simply have  $s_b^j = s_a^j = 1$  while for the forward contract  $s_b^j = s_a^j = 0$ . The quantities available at the best bid and ask quotes will be denoted by  $q_b^j$  and  $q_a^j$ , respectively. This means that the position  $x^j$  we take in asset  $j$  has to lie in the interval  $[q_b^j, q_a^j]$ . For example, the quotes for the forward contract in Table 2 mean that  $q_a^j = 377$  while  $q_b^j = -258$ .

We shall denote the payout of holding  $x^j$  units of asset  $j \in J$  by  $P^j(x^j)$ . The functions  $P^j$  are given in Table 1. We model the value  $X_T$  of the underlying at maturity as a random variable so that, in the case of forwards and the options,  $P^j(x^j)$  will be random as well. We allow the liability payment  $c$  to be random but assume that it only depends on the value of the underlying at maturity.

Modelling our risk preferences with expected utility, the portfolio optimization problem can be written as

$$\begin{aligned} & \text{minimize} && Ev(c - \sum_{j \in J} P^j(x^j)) \quad \text{over } x \in D \\ & \text{subject to} && \sum_{j \in J} S_0^j(x^j) \leq w, \end{aligned} \quad (P)$$

where

$$D := \prod_{j \in J} [q_b^j, q_a^j] \quad (3.2)$$

is the set of feasible portfolios,  $E$  denotes the expectation and  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a *loss function* describing the investor's risk preferences; see e.g. (FS11 Section 4.9). Loss functions  $v$  correspond to utility functions  $u$  with  $v(c) := -u(-c)$ . The argument of  $v$  is the unhedged part of the claim  $c$ .

Instead of expected utility, one could of course describe risk preferences with more general functions of random variables. What is important for the numerical computations below is that we can approximate the function with integration quadratures. As to the constraints, one could also include various margin requirements in the specification of the set  $D$ .

It is clear that problem (P) is highly subjective. Its optimum value and solutions depend on our

- (1) financial position described by the initial cash  $w$  and liability  $c$ ,
- (2) views on the underlying  $X_T$  described by the probabilistic model,
- (3) our risk preferences described by the loss function  $v$ .

The dependence will be studied numerically in the following sections. In pricing of contingent claims, the subjective factors will be reflected in the prices at which we are willing to trade the claims. Making the dependencies explicit is one of the

main advantages of the indifference pricing approach. Indeed, the subjectivity is the driving force behind trading in practice but it is neglected e.g. by the traditional risk neutral pricing models.

Another important feature of (P) is that it is a *convex optimization problem* as soon as the loss function  $v$  is convex which simply means that we are risk averse. Convexity is crucial in numerical solution of (P) as well as in the mathematical analysis of the indifference prices; see e.g. (Pen14).

#### 4. Numerical portfolio optimization

The first challenge in the numerical solution of problem (P) is that the objective is given in terms of an integral which, in general, does not allow for closed form expressions that could be treated by numerical optimization routines. However, in applications where the liability  $c$  only depends on the value of the underlying at maturity, the integral is one-dimensional which can be approximated well with integration quadratures. This will be the case in the applications below where we study pricing and hedging of claims contingent on the underlying price at maturity. As long as the quadrature has positive weights, the approximate objective will be a finite sum of convex functions of the portfolio vector  $x$ . In the computations below, we shall approximate the expectation by the Gauss-Legendre quadrature between each consecutive strikes.

We shall reformulate the budget constraint as two linear inequality constraints by writing the position in each asset as the sum of the long and the short position. That is,  $x^j = x_+^j - x_-^j$ , where both  $x_+^j$  and  $x_-^j$  are constrained to be positive. This results in an inequality constrained convex optimization problem with the objective and constraints represented by smooth functions. Optimal solutions will automatically have either  $x_+^j = 0$  or  $x_-^j = 0$  since, for the options, the ask-price is strictly higher than the bid-price, while for the forward and cash, the payoff of one unit of a long position is strictly smaller than for a unit of a short position. The resulting problem has 884 variables and 1769 constraints.

The problem is solved with the interior-point solver of MOSEK (ApS15) which is suitable for large-scale convex optimisation problems. To set up an instance of the optimization problem in MATLAB takes on average 11.20 seconds and its solution with MOSEK, 4.30 seconds on a PC with Intel(R) Core(TM) i5-4690 CPU @ 3.50GHz processor and 8.00 GB memory.

##### 4.1. Quotes, views and preferences

We used quotes for S&P500 index options with maturity 17 June 2016. The quotes were obtained from Bloomberg on 8 April 2016 at 2:55:00PM when the value of S&P500 index was 2056.32. The available quantities at the best quotes are given in terms of lot sizes which are 50 for forwards and 100 for options. The lending and borrowing rates are 0.0043 and 0.03, respectively, which correspond to the 1-month LIBOR rate and the borrowing rate of Yorkshire bank that offered the most

generous rate at the time.

As a base case, we modelled the logarithm of the S&P index at maturity with the Student t-distribution with the scale parameter  $\sigma$  and degrees of freedom  $\nu$  estimated from 25 years of historical daily data. The mean  $\mu$  was set to zero. The effect of varying the parameters will be studied later on.

$\mu$	$\sigma$	$\nu$
0.0000	0.0554	4.8355

Table 3. The parameters for the Student t-distribution used to model the index value at maturity.

As for the loss function  $v$  in the objective, various alternatives could be used but for simplicity and ease of computations, we used the loss exponential loss function

$$v(c) = e^{\lambda c/w}, \quad (4.1)$$

where  $w$  is the initial wealth and  $\lambda > 0$  is the risk aversion parameter. In other words, the risk preferences are described by exponential utility. It should be noted that, in general, the net position at maturity can take both positive as well as negative values which prevents the use of utility functions with constant relative risk aversion. The initial wealth  $w$  used in the examples was  $w = 100,000\text{USD}$  and the claim  $c$  was assumed to be constant zero.

#### 4.2. *The results*

Figure 1 illustrates the optimized portfolios obtained with two different risk aversions,  $\lambda = 2$  (blue line) and  $\lambda = 6$  (red line). The bottom panels represent the optimal portfolios with the bars corresponding to the optimal positions in the assets. The top left plots the corresponding payoffs as functions of the index at maturity and the top right plots the kernel density estimates (computed using 10,000,000 simulated values of the index at maturity) of the payoff distributions. As expected, higher risk aversion results in a payoff distribution with a thinner left tail. Increasing the risk aversion also results in reduced quantities in the optimal portfolio compared with the portfolio of a less risk averse agent.

An interesting feature of the optimal portfolios is that they are sparse in that out of the more than 400 quoted options, the optimal portfolio has nonzero positions in less than 10 options. This is explained by the spreads between the quotes bid- and ask-prices. To illustrate this further, we repeated the optimization with risk aversion  $\lambda = 2$  by optimizing two variants of the problem. In the first one, we increased the bid-ask spread by adding a 10% transaction cost on all trades and in the second, we set both the bid- and ask-prices equal to mid-prices. The results are illustrated in Figure 2. The addition of the transaction cost made the optimal portfolio only slightly sparser while removal of the bid-ask spread had a dramatic effect by giving a portfolio that takes large positions in almost all the quoted options. For many

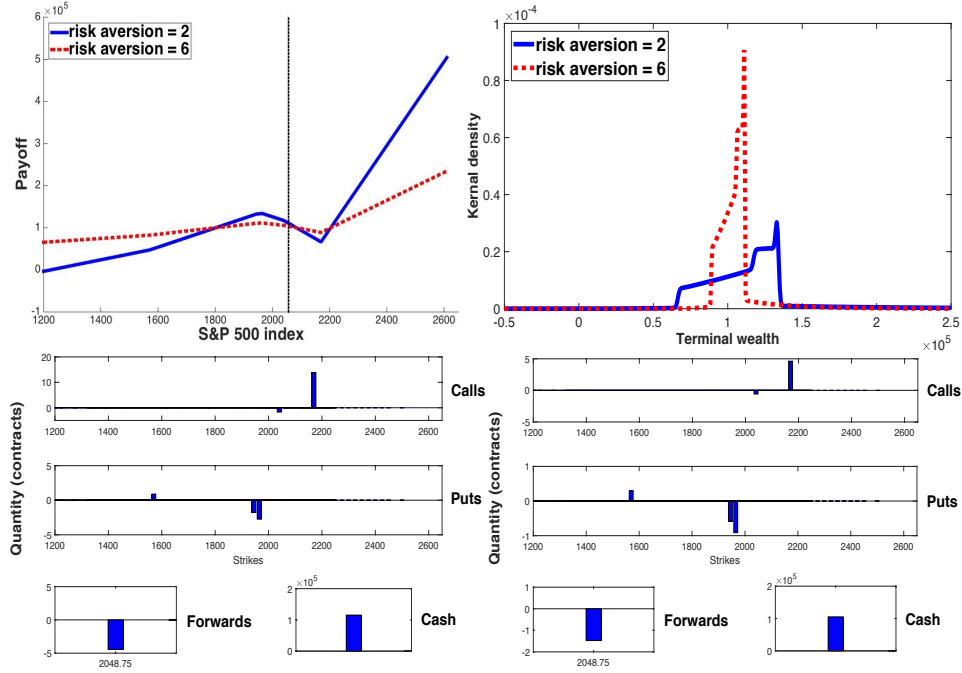


Fig. 1. The optimal portfolios obtained with risk aversions  $\lambda = 2$  and  $\lambda = 6$ , respectively (bottom), the payoffs of the optimal portfolios as functions of the index at maturity (top-left) and the kernel-density estimates of the payoff distributions of the optimal portfolios (top-right)

Base case	-2.1499
$\sigma = 0.40$	-3.5121
$\nu = 20$	-2.2339

Table 4. Logarithms of the objective values corresponding to the three different models of the underlying

options, it was optimal to take maximal positions allowed by the available bid/ask quantities.

To study the effect of views on the optimal portfolio, we reoptimized the portfolio after changing the parameters of the underlying  $t$ -distribution. The risk aversion was kept at  $\lambda = 2$ . Figure 3 plots the payouts of the optimal portfolios in three cases. The first one is the base case already presented in Figure 1. The second is obtained by increasing the scale parameter  $\sigma$  to 0.40 and the third one by increasing the degrees of freedom  $\nu$  to 20. As expected, increasing  $\sigma$  results in a portfolio that gives higher payouts further in the tails (a straddle) while  $\nu = 20$  gives essentially a Gaussian distribution with thinner tails so the optimal portfolio has higher payouts near the median at the expense of lower payoffs in the tails.



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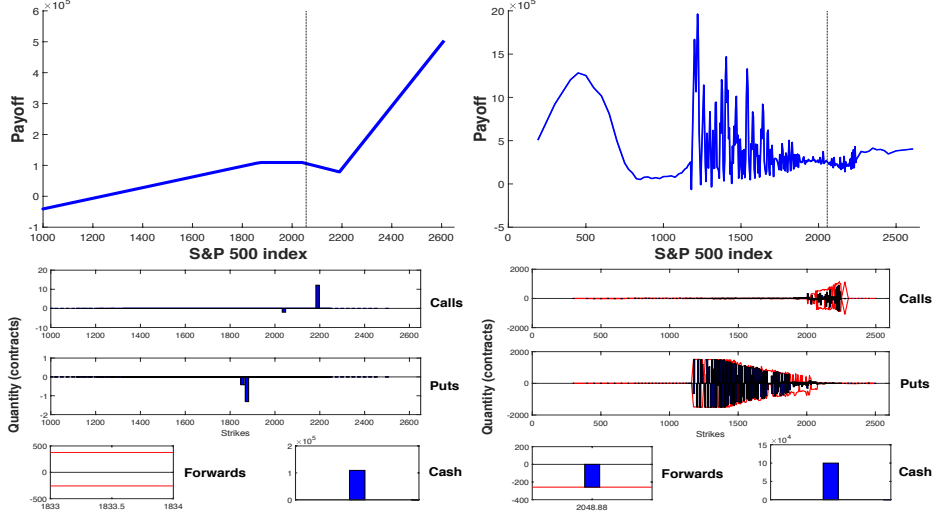


Fig. 2. The payoffs and optimal portfolios when an additional 10% transaction cost is added to all trades (left) and when the bid-ask spread is ignored by setting both bid- and ask-prices equal to the mid-price (right). The red lines represent the available quantities at the best quotes.

The logarithms of the objective values obtained with the three models of the underlying in Figure 3 are given in Table 4. The logarithm of the expected exponential utility is known as the *entropic risk measure*; see e.g. (FS11). We see that the highest objective value is obtained with in the base case where the model parameters are estimated from historical data. An explanation of this could be that the option prices used in the model correspond to the market participants' views of the future behaviour of the underlying. If we use a model that is “inconsistent” with these prices, the option prices appear to offer profitable trading opportunities.

To explore this phenomenon more systematically, we repeated the optimization in the Gaussian case with  $\nu = \infty$  and the mean  $\mu$  and volatility  $\sigma$  ranging over intervals. Figure 4 plots the corresponding logarithmic objective value, i.e. the entropic risk measure as a function of  $\mu$  and  $\sigma$ . The risk seems to be concave as a function of  $(\mu, \sigma)$  with the maximum around  $(\mu, \sigma) = (-0.05, 0.08)$ . The maximum value is  $-2.289$ .

## 5. Indifference pricing

We shall denote the optimum value of (P) by

$$\varphi(w, c) := \inf \{ Ev(c - \sum_{j \in J} P^j(x^j)) \mid x \in D, \sum_{j \in J} S_0^j(x^j) \leq w \}. \quad (5.1)$$

For an agent with financial position of  $\bar{w}$  units of initial cash and a liability to delivered a random claim  $\bar{c}$  at time  $t = 1$ , the *indifference price* for selling a claim

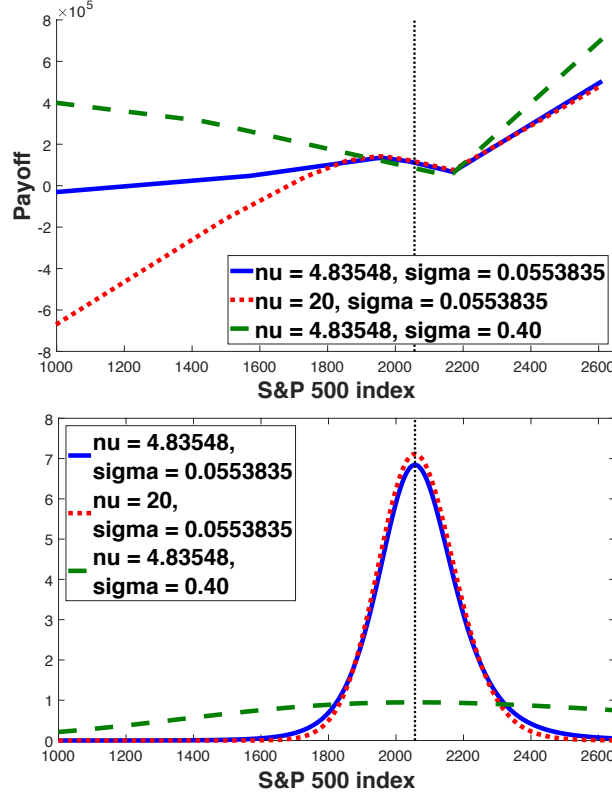


Fig. 3. Distributions of the underlying (bottom) and optimal payoffs (top) in the base case (solid line),  $\nu = 20$  (dotted) and  $\sigma = 0.40$  (dashed). All other model parameters were unchanged.

$c$  is given by

$$\pi_s(\bar{w}, \bar{c}; c) := \inf\{w \mid \varphi(\bar{w} + w, \bar{c} + c) \leq \varphi(\bar{w}, \bar{c})\}. \quad (5.2)$$

This is the minimum price at which the agent could sell the claim  $c$  without worsening her financial position as measured by the optimum value of (P). Analogously, the indifference price for buying  $c$  is given by

$$\pi_b(\bar{w}, \bar{c}; c) := \sup\{w \mid \varphi(\bar{w} - w, \bar{c} - c) \leq \varphi(\bar{w}, \bar{c})\}. \quad (5.3)$$

We have

$$\pi_b(\bar{w}, \bar{c}; c) \leq \pi_s(\bar{w}, \bar{c}; c) \quad (5.4)$$

as soon as  $\pi_s(\bar{w}, \bar{c}; 0) = 0$ . Indeed, it is easily checked that the function  $c \mapsto \pi_s(\bar{w}, \bar{c}; c)$  is convex so

$$\pi_s(\bar{w}, \bar{c}; 0) \leq \frac{1}{2}\pi_s(\bar{w}, \bar{c}; c) + \frac{1}{2}\pi_s(\bar{w}, \bar{c}; -c) \quad (5.5)$$

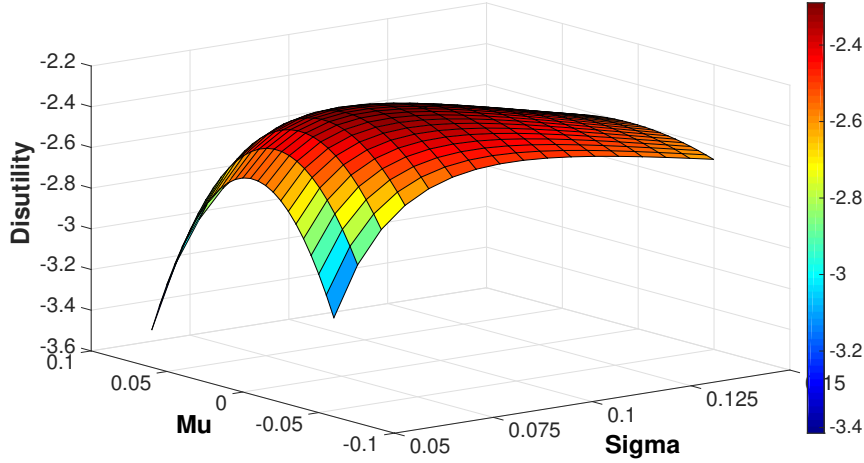


Fig. 4. The entropic risk of the optimal portfolios as a function of the mean  $\mu$  and volatility  $\sigma$  when  $\nu = \infty$

while  $\pi_s(\bar{w}, \bar{c}; -c) = -\pi_b(\bar{w}, \bar{c}; c)$ , by definition.

We shall compare the indifference prices with the superhedging and subhedging costs defined for a claim  $c$  by

$$\pi_{\sup}(c) := \inf\left\{\sum_{j \in J} S_0^j(x^j) \mid x \in D, \sum_{j \in J} P^j(x^j) - c \geq 0 \text{ } P\text{-a.s.}\right\}, \quad (5.6)$$

$$\pi_{\inf}(c) := \sup\left\{-\sum_{j \in J} S_0^j(x^j) \mid x \in D, \sum_{j \in J} P^j(x^j) + c \geq 0 \text{ } P\text{-a.s.}\right\}. \quad (5.7)$$

The superhedging cost is the least cost of a superhedging portfolio while the subhedging cost is the greatest revenue one could get by entering position that superhedges the negative of  $c$ . Whereas the indifference prices of a claim depend on our financial position, views and risk preferences described by  $(w, c)$ ,  $P$  and  $v$ , respectively, the superhedging and subhedging costs are independent of such subjective factors. In complete markets, the sub- and superhedging costs are equal for all claims  $c$  but, in general, the superhedging and subhedging costs are too wide apart to be considered as competitive quotes for a claim.

In situations where the quantities available at the best quotes are large enough to be nonbinding, the indifference prices lie between the superhedging and subhedging costs. Indeed, an application of (Pen14 Theorem 4.1) to the present situation gives the following.

**Theorem 5.1.** *The function  $\pi_s(\bar{w}, \bar{c}; \cdot)$  is convex, nondecreasing and  $\pi_s(\bar{w}, \bar{c}; 0) \leq 0$ . If there are no quantity constraints (or if they are not active), then  $\pi_s(\bar{w}, \bar{c}; c) \leq$*

$\pi_{\sup}(c)$ . If in addition,  $\pi_s(\bar{w}, \bar{c}; 0) = 0$ , then

$$\pi_{\inf}(c) \leq \pi_l(\bar{w}, \bar{c}; c) \leq \pi_s(\bar{w}, \bar{c}; c) \leq \pi_{\sup}(c) \quad \forall c \in L^0$$

with equalities throughout if  $s^b = s^a$  and  $c$  is replicable.

Recall that if  $c : \mathbb{R}_+ \rightarrow \mathbb{R}$  is the difference of convex functions, then its right-derivative is of bounded variation and we have

$$c(X_T) = c(0) + c'(0)X_T + \int_0^\infty (X_T - K)^+ dc'(K). \quad (5.8)$$

This might suggest that the payout  $c$  could be replicated by a buy-and-hold portfolio of  $c(0)$  units of a zero-coupon bond,  $c'(0)$  units of the underlying and a continuum of call options weighted according to the Borel-measure associated with the BV function  $c'$ . Even if one could buy and sell options with arbitrary strikes, it is not quite realistic to trade a continuum of them. Nevertheless, assuming that quotes for all strikes exist, the replication cost of  $c$  would become

$$c(0)P_T + c'(0)X_0 + \int_0^\infty C(K)^a dc'_+(K) - \int_0^\infty C(K)^b dc'_-(K), \quad (5.9)$$

where  $c'_+$  and  $c'_-$  denote the positive and negative variations, respectively, of  $c'$  and  $C(K)^b$  and  $C(K)^a$  denote the bid- and ask-prices of a call with strike  $K$ .

Equation (5.8) could be used to design approximate replication strategies given a finite number of quotes in real markets. We shall find out that the hedges optimized for indifference pricing look quite different from what equation (5.8) would suggest. Indifference pricing looks for approximate hedges that are optimal for the given bid- and ask-quotes, risk preferences and the given probabilistic description of the underlying.

### 5.1. Numerical computation of indifference prices

The definitions of the indifference prices involve the optimum value function  $\varphi$  of problem (P) which can rarely be evaluated exactly. The definitions still make sense, however, if we replace the optimum value by the best value we are able to find numerically. Besides the financial position, future views and risk preferences of an agent, the indifference prices then depend also on the agent's expertise in portfolio optimization. In computations below, we shall replace  $\varphi$  by the approximate value we find with the numerical techniques described in Section 4.1. The evaluation of the indifference prices then come down to a one-dimensional search over  $w$ . This can be done numerically by a line-search algorithm. In the numerical illustrations below, we used a simple bisection method.

The computation of the superhedging and subhedging costs come down to solving linear programming problems where the constraints require the terminal position of the agent to be nonnegative in every scenario; see (KKP05). In the context of put and call options, the constraint can be written in terms of finitely many linear inequality constraints since we know that the hedging error will be linear between consecutive strike prices.

### 5.2. Pricing exotic options

We illustrate indifference pricing using the optimization model of Section 3 in the pricing of three “exotic” options namely, a digital option with payoff

$$c(X_T) = \begin{cases} 10,000 & \text{if } X_T \geq K, \\ 0 & \text{if } X_T < K, \end{cases} \quad (5.10)$$

a “quadratic forward” with

$$c(X_T) = |X_T - K|^2 \quad (5.11)$$

and a “log-forward” with

$$c(X_T) = 100,000 \ln(K/X_T), \quad (5.12)$$

all with strike  $K = 2050$ . Log-forwards arise e.g. in hedging of variance swaps; see e.g. (CM98). To compare with a simpler option, we also price a European call option with the same strike. To make the last case nontrivial, we remove the call from the set of hedging instruments.

We compute the indifference selling prices assuming that  $\bar{w} = 100,000$  and  $\bar{c} = 0$ , that is, assuming the agent has initial position consisting only of 100,000 units of cash. The indifference prices together with the superhedging and subhedging costs are given in Table 5. Clearly, superhedging the quadratic and log-forwards with the given hedging instruments against all positive values of  $X_T$  is impossible, so we shall require superhedging only on the interval  $[100, 5000]$ .

Claim	subhedging	buying price	selling price	superhedging
call	51.2333	51.7338	51.7399	53.0483
digital call	5280.00	6082.35	6160.65	6885.71
quadratic forward	20383.68	20979.84	22044.92	24542.01
log-forward	322.28	358.49	404.67	499.69

Table 5. Indifference prices, together with superhedging and subhedging costs.

Figures 5–8 illustrate the corresponding hedging strategies. Each figure gives the optimal portfolio before and after selling the option together with the payout of the “hedging portfolio” as a function of the underlying at maturity. The *hedging portfolio* is defined as the difference  $x - \bar{x}$ , where  $\bar{x}$  and  $x$  are the optimal portfolios before and after the sale of the option.

### 5.3. Sensitivities

This section studies the sensitivities of the indifference prices with respect to some of the model parameters. Figure 9 plots indifference prices of a call option with strike 2000 as functions of the scale parameter  $\sigma$ . Since we model the underlying with

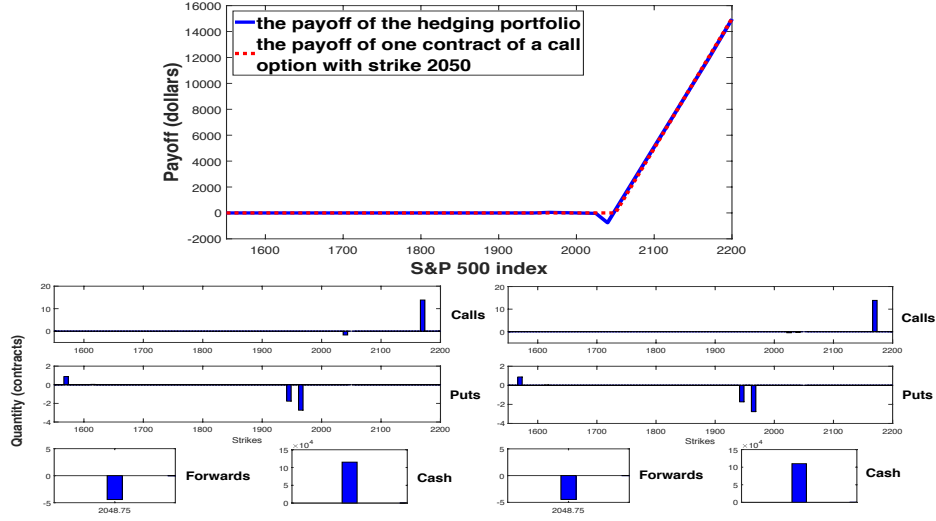


Fig. 5. Optimal portfolios before (bottom left) and after (bottom right) the sale of a call option. The top panel gives the payoff of the hedging portfolio (solid line) together with the payoff of the claim being priced (dotted line).

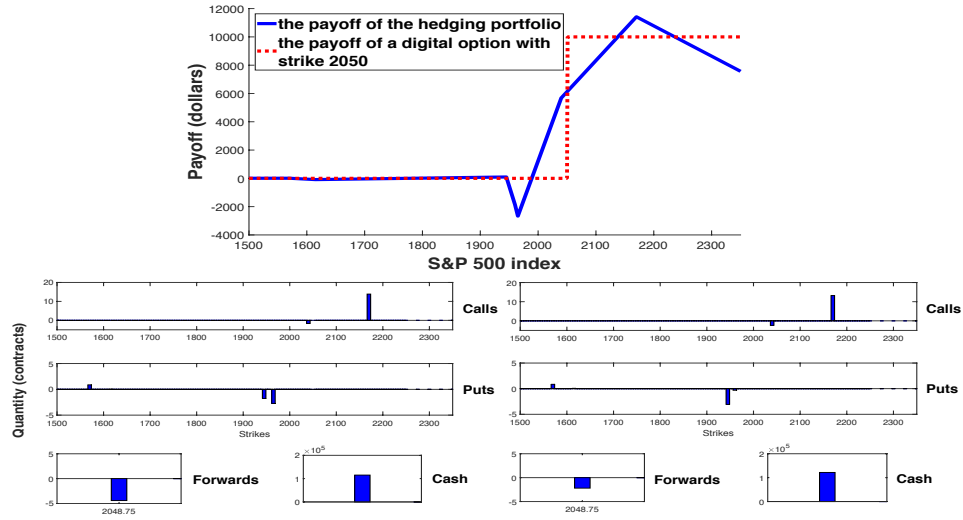


Fig. 6. Optimal portfolios before (bottom left) and after (bottom right) the sale of a digital option. The top panel gives the payoff of the hedging portfolio (solid line) together with the payoff of the claim being priced (dotted line).

t-distributions, the variance of the log-price is  $\sigma^2\nu/(\nu-2)$ . Again, we have removed the call being priced from the set of hedging instruments when computing the prices.

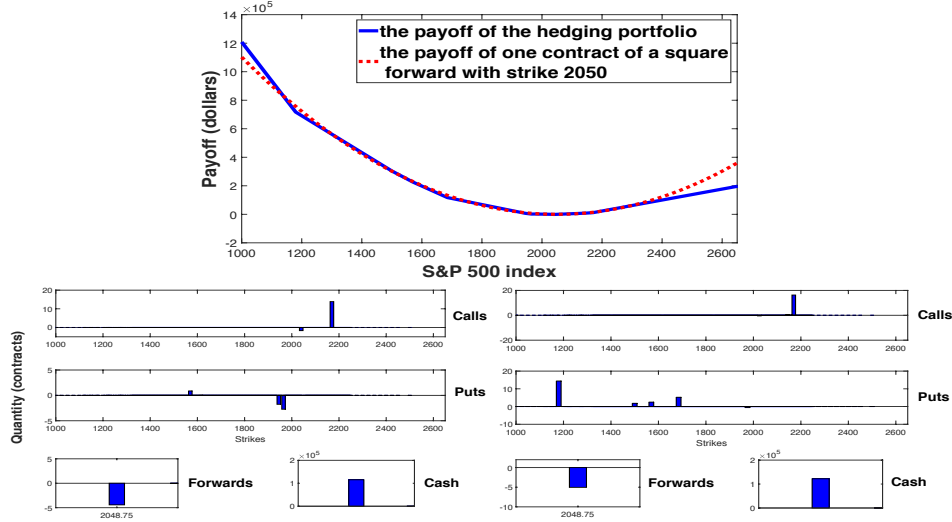
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Fig. 7. Optimal portfolios before (bottom left) and after (bottom right) the sale of a quadratic forward. The top panel gives the payoff of the hedging portfolio (solid line) together with the payoff of the claim being priced (dotted line).

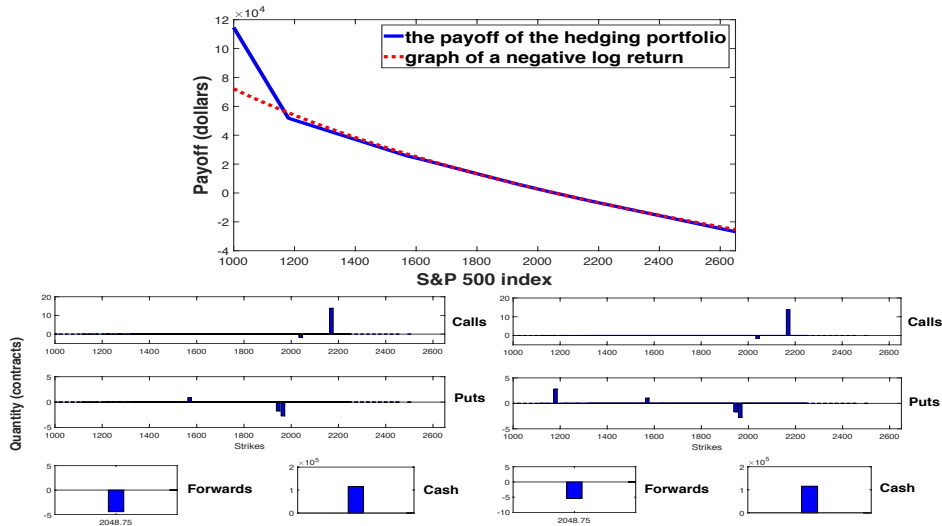


Fig. 8. Optimal portfolios before (bottom left) and after (bottom right) the sale of a log-forward. The top panel gives the payoff of the hedging portfolio (solid line) together with the payoff of the claim being priced (dotted line).

Instead of being monotone, the indifference prices achieve their minimums when  $\sigma$  is close to its historical estimate of 0.0554. The implied volatility computed with

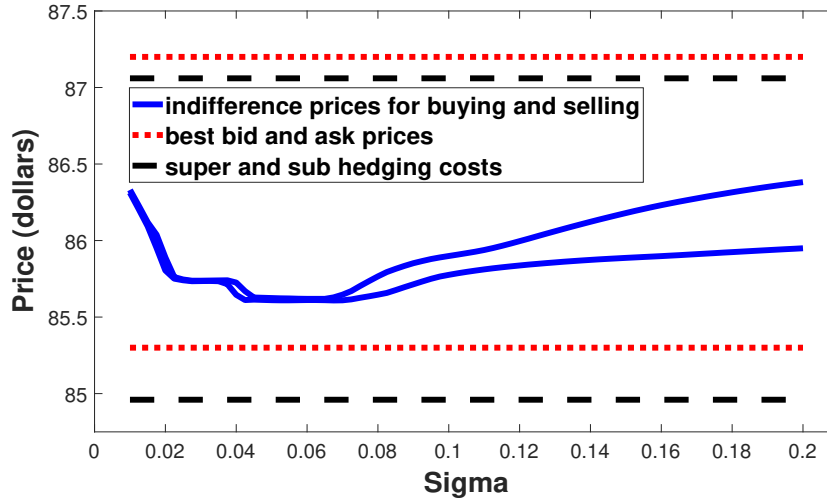


Fig. 9. Indifference prices as functions of the scale parameter  $\sigma$ . The dotted lines give the best bid and ask quotes while the dashed lines give the superhedging and subhedging costs.

the classical Black–Scholes model from the mid-quote of the call is 0.1478.

Figure 10 plots the prices as functions of the location parameter  $\mu$ . Unlike the classical risk neutral valuations, indifference prices do depend on the growth assumptions on the underlying, in general. The prices are lowest when  $\mu$  is near zero or slightly negative. One might think that this reflects the “market’s view” encoded in the price quotes.

Figure 11 plots the indifference prices as functions of the risk aversion parameter  $\lambda$ . As the risk aversion increases, the gap between the indifference prices widens. The indifference price for selling a call option is more sensitive to the risk aversion. This seems quite natural as shorting a call results in unbounded downside risk unless the call is superhedged.

Figure 12 illustrates the dependence of the indifference prices on an agent’s initial position. While in earlier cases, the agent’s initial position was assumed to consist only of cash, in this case, we consider an agent with both cash and call options of the same type as the one being priced. Figure 12 plots the indifference prices as functions of the number of call options the agent holds before the trade. As one might expect, an agent who already has exposure to the option would assign a higher price to the option. A seller would increase her exposure to the option payout while for a buyer, the option would be a natural hedge and thus worth paying a higher price for.

To illustrate the nonlinearity of the indifference prices as functions of the claim, we computed the prices for different multiples  $M$  of the call. Figure 13 plots the indifference prices per option as functions of the multiplier  $M$ . The figure plots



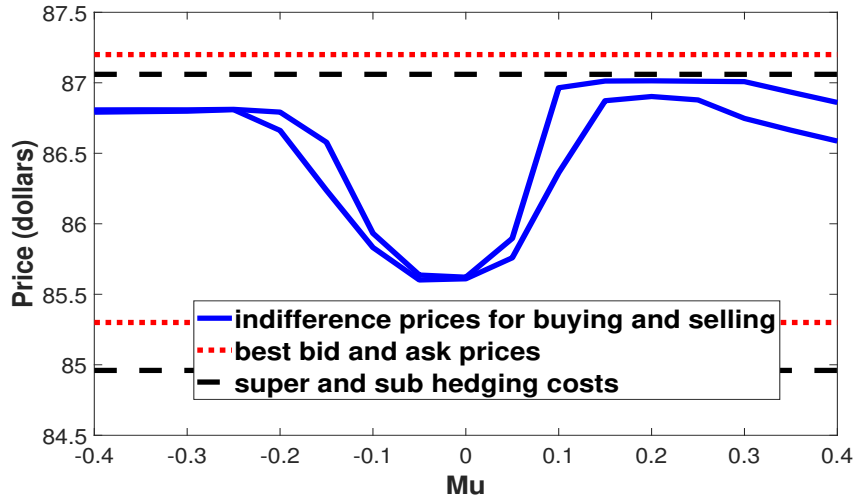


Fig. 10. Indifference prices as functions of the location parameter  $\mu$ . The dotted lines give the best bid and ask quotes while the dashed lines give the superhedging and subhedging costs.

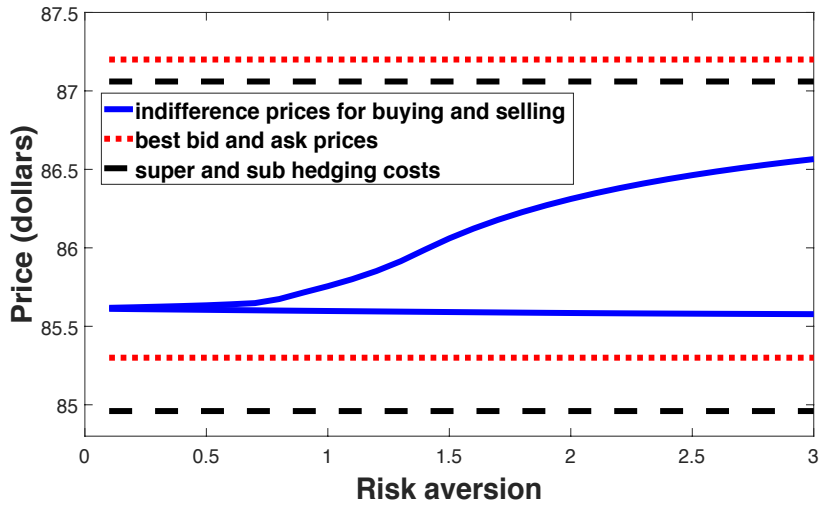


Fig. 11. Indifference prices of a call option with strike 2000 as functions of risk aversion  $\lambda$ .

the indifference prices also in a market model where the best quotes are assumed to come with unlimited quantities. As the multiplier  $M$  increases, the quantity constraints become binding thus worsening the prices.

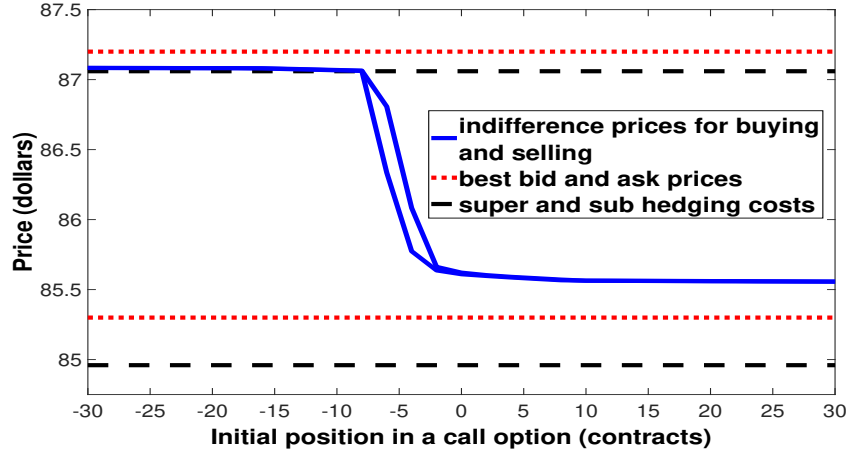


Fig. 12. Indifference prices of a call option with strike 2000 as functions of initial position in the same call

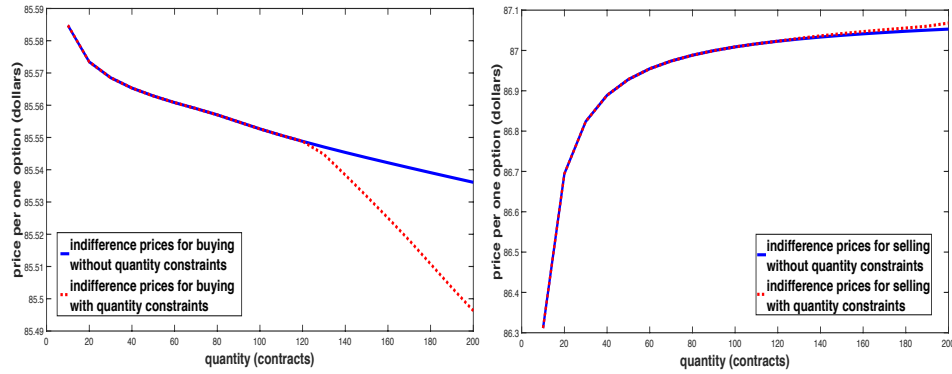


Fig. 13. Indifference prices of a call option per unit as a function of the quantity traded. Buying price on the left and selling price on the right. The solid line gives the prices when quantity constraints are ignored

## 6. Further developments

The developed computational framework should be taken merely as an illustration of some of the techniques that are available for portfolio optimization and indifference pricing in practice. The presented model could be extended in various ways. For example, it would be straightforward to include margin requirements as portfolio constraints in the model as long as the requirements are given as explicit convex constraints on the portfolio. The model could also be extended to include options with multiple maturities as well as dynamic trading strategies of the underlying and

cash. One could also study hybrid derivatives whose payouts depend on multiple underlyings. Such extensions increase the number of underlying risk factors which would need to be modelled and approximated in the numerical representation of the objective function. The number of quadrature points required to reach a given level of accuracy often grows superlinearly with the number of risk factors so the computation times are like to increase significantly in such extensions. The information based complexity of such problems is analysed in (Pen12 Section 5.2).

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